Unit 3 Polynomials
Section 1 Classify, Add, Subtract, and Multiply Polynomials

Objective: Classify polynomials by degree and number of terms. Add, subtract, or multiply polynomials.

MONOMIAL:

EX.

DEGREE OF A MONOMIAL:

Identify the degree of each monomial.

1) x^6

2) 13

3) $8x^3y$

POLYNOMIAL:

EX.

DEGREE OF A POLYNOMIAL:

Identify the degree of each polynomial.

4) $x^4 + 4x^3 + 2x^2$

5) $4x^3 - 3$

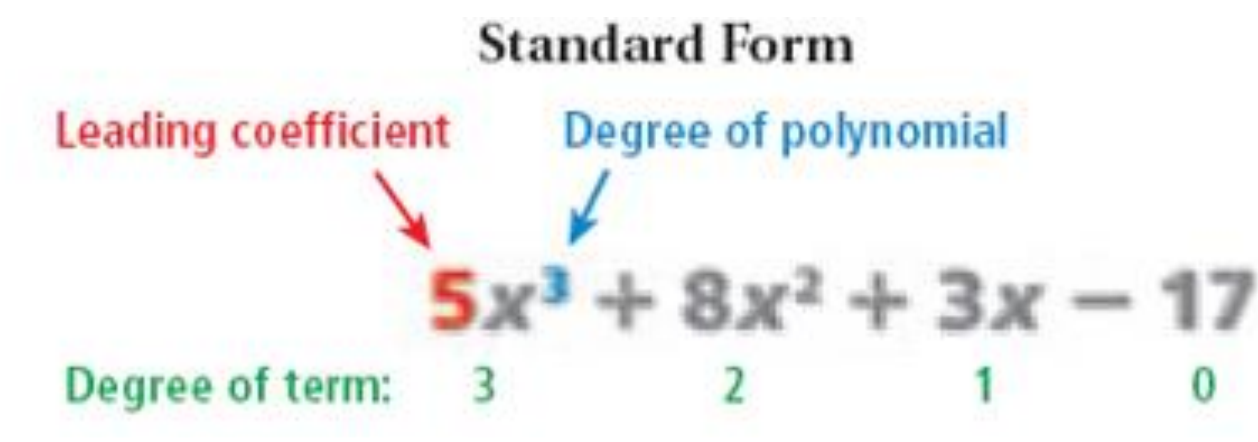
6) $8x^3y^4 + 7x^2y^3 + 9xy^4$

STANDARD FORM OF A POLYNOMIAL:

EX.

LEADING COEFFICIENT:

EX.



A polynomial can be classified by its **<u>number of terms</u>.**

Name	# of Terms	Example
Monomial		
Binomial		
Trinomial		
Polynomial		

A polynomial can also be classified by its **<u>degree</u>.**

Name	Degree	Example
Constant		
Linear		
Quadratic		
Cubic		
Quartic		
Quintic		

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

7) $3 - 5x^2 + 4x$ Standard form: Leading Coeff: Degree: # of terms: Name:	8) $8x^5 + 1 - 3x^2 - 9x^4$ Standard form: Leading Coeff: Degree: # of terms: Name:	9) $-18x^2 + x^3$ Standard form: Leading Coeff: Degree: # of terms: Name:

Adding and Subtracting Polynomials

REMEMBER: To add or subtract polynomials, combine like terms.

Add or subtract. Write your answer in standard form.

10) $(2x^3 + 7 - x) + (5x^2 + 4 + 8x + x^3)$	11) $(4 - 2x^2) - (x^2 + 5 - x)$
12) $(-35x^2 + 6x - 11) + (7x^2 + 16x^3 - 5)$	13) $(3 - 4x^2) - (x^2 + 6 - x)$

Multiplying Polynomials

To multiply any two polynomials, use the Distributive Property and multiply each term in the second polynomial by each term in the first.

$$(x + 2)(x^2 + 4x - 3)$$

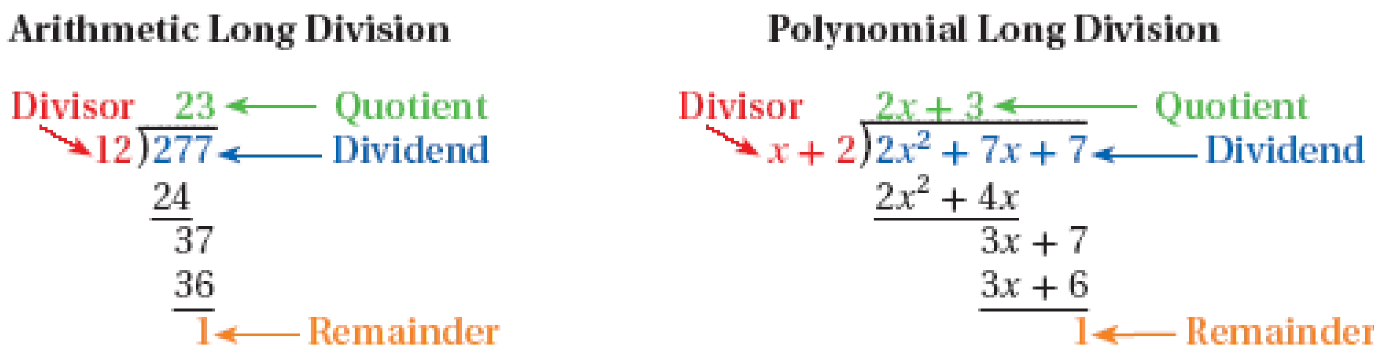
REMEMBER: $x^a * x^b = x^{a+b}$ *so* $x^3 * x^2 = x^5$

14) $5y^2(y^2 + 4)$	15) $x^2y(8y^3 + y^2 - 27y + 31)$
16) $(x + 7)(x - 6)$	17) $(x + 4)(3x - 5)$
18) $(a + 4b)^2$	19) $(a - 3)(2 - 7a + a^2)$
20) $(y - 3)(y^2 - 4a + 9)$	21) $(x^2 - 4x + 1)(x^2 + 5x - 2)$

Unit 3 Polynomials
Section 2 Division of Polynomials

Objective: Divide polynomial expressions by long division and synthetic division when appropriate.

Polynomial long division is a method for dividing a polynomial by another polynomials of a lower degree. It is very similar to dividing numbers.



Divide using long division.

1) $(15x^2 + 8x - 12) \div (3x + 1)$

2) $(-y^2 + 2y^3 + 25) \div (y - 3)$

3) $(9x^5 - 69x^4 + 21x^3 - 57x^2 - 33x - 8) \div (9x + 3)$

<u>Synthetic division</u> is a shorthand method of dividing a polynomial by a linear binomial by using only the coefficients. For synthetic division to work, the polynomial must be written in standard form, using 0 and a coefficient for any missing terms, and the divisor must be in the form ($x - a$).

Synthetic Division Method

Divide $(2x^2 + 7x + 9) \div (x + 2)$ by using synthetic division.

WORDS	NUMBERS
Step 1 Write the coefficients of the dividend, 2, 7, and 9. In the upper left corner, write the value of *a* for the divisor ($x - a$). So $a = -2$. Copy the first coefficient in the dividend below the horizontal bar.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & & & \\ \hline & 2 & & \end{array}$
Step 2 Multiply the first coefficient by the divisor, and write the product under the next coefficient. Add the numbers in the new column.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & & -4 & \\ \hline & 2 & 3 & \end{array}$
Repeat Step 2 until additions have been completed in all columns. Draw a box around the last sum.	$\begin{array}{r\|rrr} -2 & 2 & 7 & 9 \\ & & -4 & -6 \\ \hline & 2 & 3 & \boxed{3} \end{array}$
Step 3 The quotient is represented by the numbers below the horizontal bar. The boxed number is the **remainder**. The others are the coefficients of the polynomial quotient, in order of decreasing degree.	$= 2x + 3 + \dfrac{3}{x + 2}$

Divide using synthetic division.

1) $2x^3 + 9x^2 + x - 12$ **by** $x + 4$

2) $(3x^4 - x^3 + 5x - 1) \div (x + 2)$

3) $\dfrac{2x^3 + 4x + 5}{x - 1}$

4) $\dfrac{4t^3 - 30t^2 + 8t + 15}{2t + 1}$

Unit 3 Polynomials
Section 3 Factoring Polynomials

Objective: Factor polynomial expressions using factoring by grouping and sum or difference of cubes.

Just as there is a special rule for factoring the difference of two squares, there are special rules for factoring the sum or difference of two cubes.

Factoring the Sum and the Difference of Two Cubes

METHOD	ALGEBRA
Sum of two cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of two cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Factor

1) $x^3 + 8$

2) $x^3 - 216$

3) $125x^3 - 1$

4) $64x^3 - 27$

Factoring Polynomials Completely
Any Polynomial: Look for the Greatest Common Factor (GCF)
Binomial: Look for Difference of Squares or Sum and Difference of Cubes
Trinomial: Look to factor the trinomial (Guess and Check, FISH, X)
4 or More Terms: Factor by Grouping

Factor

5) $7x^2 - 54x - 81$

6) $25x^2 - 4$

7) $4p^3 - 3p^2 - 20p + 15$

8) $5p^2 - 20p$

9) $5n^3 + 60n^2 + 160n$

10) $5a^2 + 9a - 2$

11) $14m^3 - 7m^2 + 2 - 4m$

12) $x^2 - 64$

13) $16u^3 + 250$

14) $8p^3 - 40p^2 - p + 5$

Unit 3 Polynomials
Section 4 Remainder Theorem and Factor Theorem

Objective: Apply the Remainder Theorem and Factor Theorem

You can use synthetic division to evaluate polynomials. This process is called _____________________________. The process of synthetic substitution is exactly the same as the process of synthetic division, but the final answer is interpreted differently, as described by the Remainder Theorem.

Remainder Theorem

THEOREM	EXAMPLE
If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.	Divide $x^3 - 4x^2 + 5x + 1$ by $x - 3$. $\begin{array}{r\|rrrr} 3 & 1 & -4 & 5 & 1 \\ & & 3 & -3 & 6 \\ \hline & 1 & -1 & 2 & \boxed{7} \end{array}$ $P(3) = 7$

$P(x) = x^3 + 3x^2 + 4$ for $x = -3$	$P(x) = 2x^3 + 5x^2 - x + 7$ for $x = 2$
$P(x) = 6x^4 - 25x^3 - 3x + 5$ for $x = -\frac{1}{3}$	$P(x) = x^3 + 3x^2 - 6$ for $x = 5$

Recall that if a number is divided by any of its factors, the remainder is 0. Likewise, if a polynomial is divided by any of its factors, the remainder is 0.

The Remainder Theorem states that if a polynomial is divided by ($x - a$), the remainder is the value of the function at a. So, if ($x - a$) is a factor of $P(x)$, then $P(a) = 0$.

Factor Theorem

THEOREM	EXAMPLE
For any polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.	Because $P(1) = 1^2 - 1 = 0$, $(x - 1)$ is a factor of $P(x) = x^2 - 1$.

Tell whether or not the binomial is a factor of the polynomial.

$x + 1$; $P(x) = x^2 - 3x + 1$	$x + 2$; $P(x) = 3x^4 + 6x^3 - 5x - 10$
$x + 2$; $P(x) = x^3 + 2x^2 - x - 2$	$x - 1$; $P(x) = 3x^2 - 2x + 5$

Unit 3 Polynomials
Section 5 Finding Real Roots of Polynomial Equations

Objective: Find roots of polynomial equations using the Fundamental Theorem of Algebra, Rational Root Theorem, and Conjugate Root Theorem.

Solve each polynomial equation by factoring.

1. $2x^6 - 10x^5 - 12x^4 = 0$

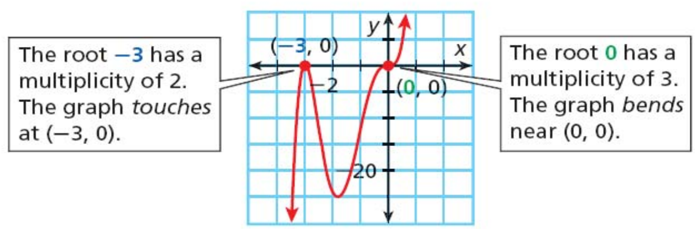
2. $x^3 - 2x^2 - 25x = -50$

A polynomial equation can sometimes have a factor that appears more than once. This creates a ***multiple root***.

For example,
$3x^5 + 18x^4 + 27x^3 = 0$
$3x^3(x + 3)(x + 3) = 0$

Has two multiple roots, 0 and –3. In fact, the root 0 is a factor three times since $3x^3=0$ and the root – 3 is a factor twice.

The **multiplicity** of root r is the number of times that $x - r$ is a factor of $P(x)$. When a real root has even multiplicity, the graph of $y = P(x)$ touches the x-axis but does not cross it. When a real root has odd multiplicity greater than 1, the graph "bends" as it crosses the x-axis.



You cannot always determine the multiplicity of a root from a graph. It is easiest to determine multiplicity when the polynomial is in factored form.

Identify the roots of each equation. State the multiplicity of each root.

3. $x^3 - 3x^2 - 9x + 27 = 0$

4. $2x^6 - 22x^5 + 36x^4 = 0$

Looking back at numbers 3 and 4 are there any connections we can make between the number of solutions and the degree of the polynomial?

The Fundamental Theorem of Algebra: Every polynomial function of degree n ≥ 1 has at least one zero, where a zero may be a complex number.
Corollary: Every polynomial function of degree n ≥ 1 has exactly *n* zeros, including multiplicities.

State the number of complex roots of each polynomial equation.

5. $4x^5 - 4x + 1 = 0$

6. $x^8 - 9x^7 + 4x^5 - 3x - 2 = 0$

RECALL

Factor Theorem

THEOREM	EXAMPLE
For any polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.	Because $P(1) = 1^2 - 1 = 0$, $(x - 1)$ is a factor of $P(x) = x^2 - 1$.

This will allow us to plug in the given solutions to get a new equation we can use to solve.

Solve the following equations given some of the solutions.

7. $0 = 2x^3 - x + 1 :\ -1 \text{ is a root}$

8. $x^4 + x^3 - 7x^2 - 9x - 18 = 0: 3 \text{ and } -3 \text{ are roots}$

9. $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0:\ 3 \text{ and } 5 \text{ are roots}$

Unit 3 Polynomials
Section 6 Irrational and Complex Roots

Objective: Find roots of polynomial equations using the Fundamental Theorem of Algebra, Rational Root Theorem, and Conjugate Root Theorem.

Irrational Root Theorem

If the polynomial $P(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

Complex Conjugate Root Theorem

If $a + bi$ is a root of a polynomial equation with real-number coefficient (a and b are real numbers and $b \neq 0$), then $a - bi$ is also a root.

A polynomial function *P(x)* with a rational coefficients has the given roots. Find the two additional roots of *P(x)* = 0.

1. $-2i \text{ and } \sqrt{10}$

2. $2 + 2i \text{ and } 2 - \sqrt{5}$

Solve each equation given the indicated root.

3. $x^3 - 3x^2 + x - 3 = 0; i$

4. $2x^3 - 3x^2 + 8x - 12 = 0; -2i$

5. $x^4 + 4x^3 - 3x^2 - 12x = 0;\ \sqrt{3}$

6. $x^4 - x^3 - 8x^2 + 2x + 12 = 0;\ \sqrt{2}$

7. $x^4 - 6x^3 + 6x^2 + 24x - 40 = 0$**;** $3+i$

8. $x^4 - 13x^3 + 55x^2 - 71x = 0; 3 - 2i$

Unit 3 Polynomials
Section 7 Solving Polynomial Equations

Objective: Find roots of polynomial equations using the Fundamental Theorem of Algebra, Rational Root Theorem, and Conjugate Root Theorem.

When we are not given any of the roots we use the ***RATIONAL ROOT THEOREM*** to find all of the **POSSIBLE** rational roots.

Rational Root Theorem

If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term of $P(x)$ and q is a factor of the leading coefficient of $P(x)$.

Example: $6x^3 + 8x^2 - 7x - 3 = 0$ has a numerator that is a factor of -3 (namely, ± 3 or ± 1) and a denominator that is a factor of 6 (namely $\pm 1, \pm 2, \pm 3$, or ± 6). So, there are 12 possible rational roots:

$$\pm\frac{1}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{1}{3}, \quad \pm\frac{1}{6}$$

Use the Rational Root Theorem to list the possible rational roots.

1. $4x^4 + 3x^2 - 1 = 0$	**2.** $2x^4 + 3x^3 - 7x^2 + 3x - 9 = 0$
3. $2x^3 - 5x^2 + 6x - 2 = 0$	

Using the Rational Root Theorem to find the possible solutions, solve the polynomial equations.

4. $4x^4 + 3x^2 - 1 = 0$	**5.** $2x^4 + 3x^3 - 7x^2 + 3x - 9 = 0$
6. $2x^3 - 5x^2 + 6x - 2 = 0$	**7.** $0 = 2x^4 + 3x3 - 17x^2 - 27x - 9$
8. $0 = 3x^3 - 7x^2 - 14x + 24$	**9.** $0 = x^4 + 3x^3 + x^2 - 12x - 20$

Unit 3 Polynomials
Section 8 Writing Polynomial Functions

Objective: Write polynomial equations when given a root or root using the Fundamental Theorem of Algebra and Conjugate Root Theorem

RECALL

The Fundamental Theorem of Algebra: Every polynomial function of degree n ≥ 1 has at least one zero, where a zero may be a complex number.
Corollary: Every polynomial function of degree n ≥ 1 has exactly *n* zeros, including multiplicities.

Irrational Root Theorem

If the polynomial $P(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

Complex Conjugate Root Theorem

If $a + bi$ is a root of a polynomial equation with real-number coefficient (a and b are real numbers and $b \neq 0$), then $a - bi$ is also a root.

Write a polynomial function of least degree with integral coefficients that has the given zeros.

1. 1, 4, -5

2. $-3, \frac{1}{2}, 5$

3. 3 (mult. of 2), – 1, 0

4. $i, 3$

5. $-3i, -5, 4$

6. $\sqrt{2}, 4, -7$

When the roots are more complex we can use the sums and differences formula to multiply them.

$$x^2 - (SUM\ OF\ THE\ TWO\ ROOTS)x + (PRODUCT\ OF\ THE\ TWO\ ROOTS)$$

7. $3 + i, 3 - i$

8. $1 - \sqrt{2}, 9$

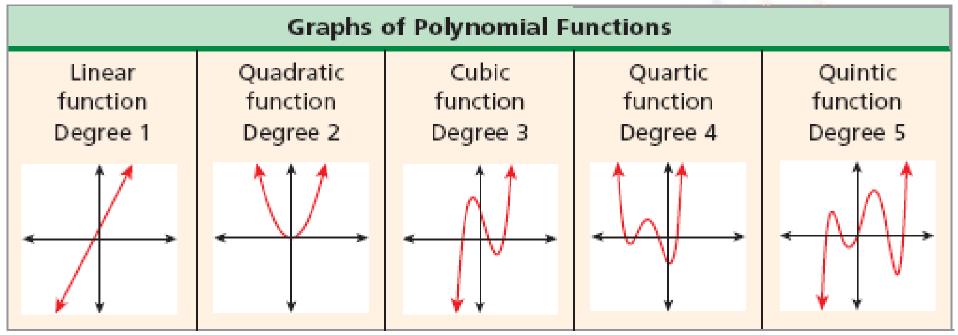
9. $2 - i, 0, -7$

10. $1 + 2i, 3 - \sqrt{3}$

Unit 3 Polynomials
Section 9 Investigating Graphs of Polynomial Functions

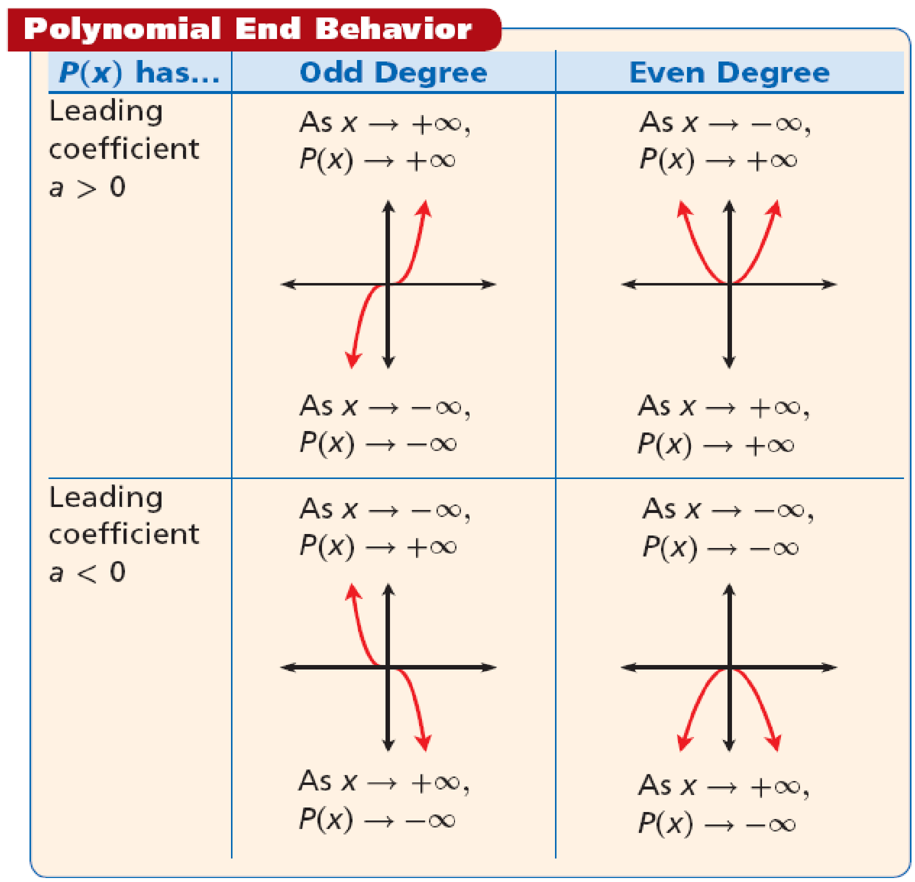
Objective: Identify the domain and range, local maxima and minima of a function.
Construct a rough sketch of a polynomial function using end behavior and roots

Polynomial functions are classified by their degree. The graphs of polynomial functions are classified by the degree of the polynomial. Each graph, based on the degree, has a distinctive shape and characteristics.



<u>End behavior</u> is a description of the values of the function as x approaches infinity ($x \rightarrow +\infty$) or negative infinity ($x \rightarrow -\infty$).

The **degree** and **leading coefficient** of a polynomial function determine its end behavior. It is helpful when you are graphing a polynomial function to know about the end behavior of the function.



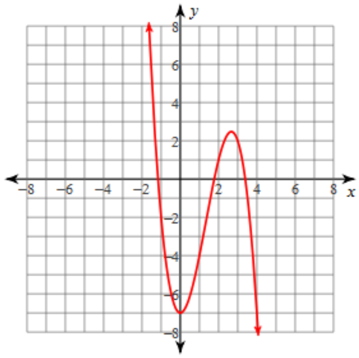
Identify the leading coefficient, degree, and end behavior.

1. $P(x) = 4x^5 + 5x^2 - 4x - 1$

2. $Q(x) = -x^4 + 6x^3 - x + 9$

Ex. 2: Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

3.



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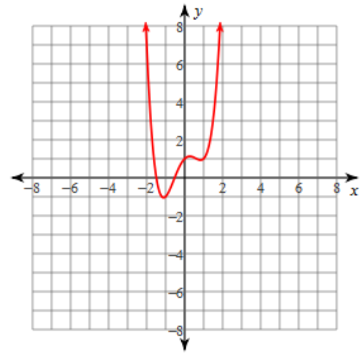

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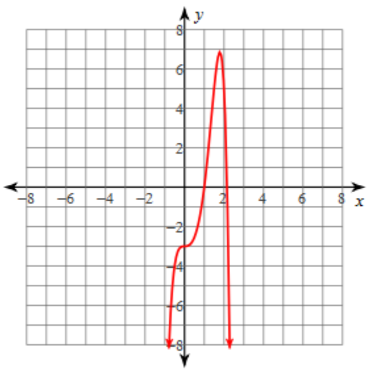

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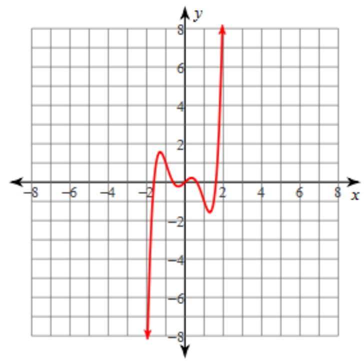


A **<u>turning point</u>** is where a graph changes from increasing to decreasing or from decreasing to increasing. A turning point corresponds to a *local maximum* or *minimum*.

Local Maxima and Minima

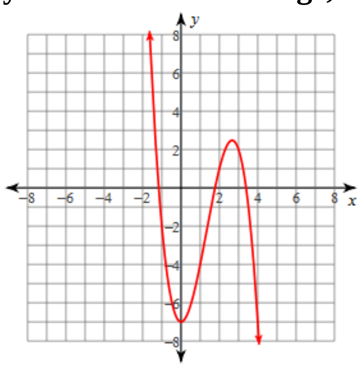
For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x-value in the interval except a.

For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x-value in the interval except a.

A polynomial function of degree n has at most $n - 1$ turning points and at most n x-intercepts. If the function has n distinct roots, then it has exactly $n - 1$ turning points and exactly n x-intercepts. You can use a graphing calculator to graph and estimate maximum and minimum values.

Identify the domain and range, local maxima and minima of a function.

7.



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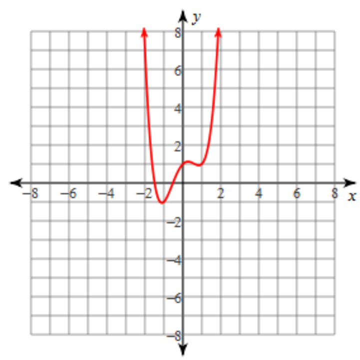

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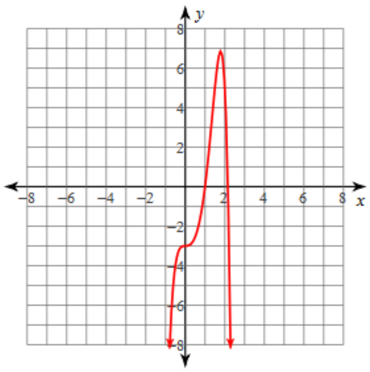

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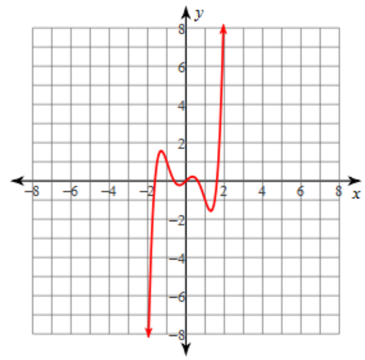


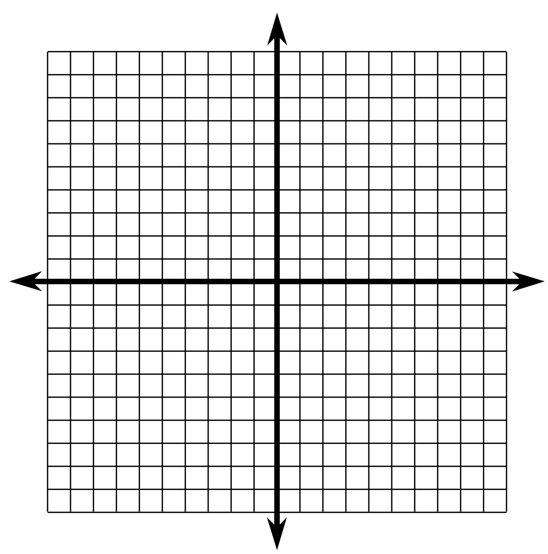
11. Graph the function. $f(x) = x^3 + 4x^2 + x - 6.$

Step 1: Identify the possible rational roots by using the Rational Root Theorem.

Step 2: Test all possible rational zeros until a zero is identified.

Step 3: Find all other zeros.

Step 4: Find the y-intercept.



Step 5: Identify end behavior.

Step 6: Find other points

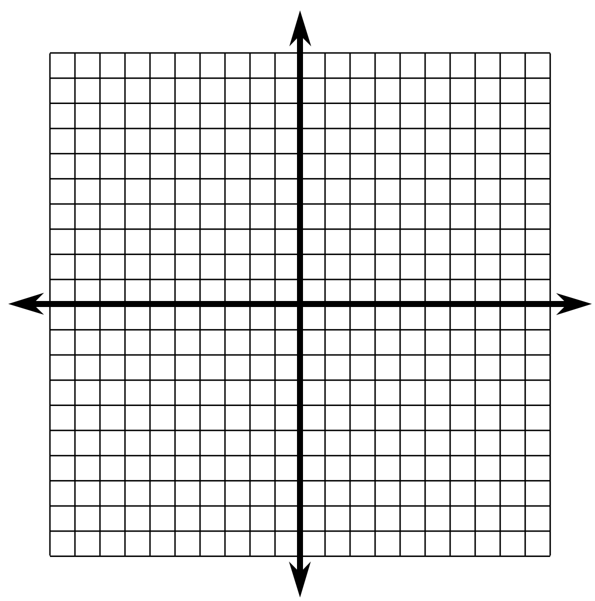
12. Graph the function. $f(x) = x^4 + x^3 - 2x^2$.

Step 1: Identify the possible rational roots by using the Rational Root Theorem.

Step 2: Test all possible rational zeros until a zero is identified.

Step 3: Find all other zeros.

Step 4: Find the y-intercept.



Step 5: Identify end behavior.

Step 6: Find other points

Unit 3 Polynomials
Section 10 Transforming Polynomial Functions

Objective: Identify transformations applied to the polynomial parent function given an equation

You can perform the same transformations on polynomial functions that you performed on our basic parent functions.

Transformations of $f(x)$

Transformation	$f(x)$ Notation	Examples	
Vertical translation	$f(x) + k$	$g(x) = x^3 + 3$ $g(x) = x^3 - 4$	3 units up 4 units down
Horizontal translation	$f(x - h)$	$g(x) = (x - 2)^3$ $g(x) = (x + 1)^3$	2 units right 1 unit left
Vertical stretch/ compression	$af(x)$	$g(x) = 6x^3$ $g(x) = \frac{1}{2}x^3$	stretch by 6 compression by $\frac{1}{2}$
Horizontal stretch/ compression	$f\left(\frac{1}{b}x\right)$	$g(x) = \left(\frac{1}{5}x\right)^3$ $g(x) = (3x)^3$	stretch by 5 compression by $\frac{1}{3}$
Reflection	$-f(x)$ $f(-x)$	$g(x) = -x^3$ $g(x) = (-x)^3$	across x-axis across y-axis

Example 1: For $f(x) = x^3 - 5$, write the rule for each function

a. $g(x) = f(x) + 7$

b. $g(x) = f(x + 2)$

Example 2: Let $f(x) = x^3 + 5x^2 - 8x + 1$. Write a function g that performs each transformation.

a. Reflect f across the x-axis

b. Reflect f across the y-axis

Example 3: Let $f(x) = 2x^4 - 6x^2 + 1$.

a. $g(x) = \frac{1}{2}f(x)$

b. $g(x) = f\left(\frac{1}{3}x\right)$

Example 4: Write a function that transforms $f(x) = 6x^3 - 3$ in each of the following ways.

a. Compress vertically by a factor of $\frac{1}{3}$, and shift 2 units right.

b. Reflect across the y-axis and shift 2 units down.

Example 5: For $f(x) = x^3 - 8$, describe the transformation.

a. $g(x) = x^3 - 2$

b. $h(x) = (x + 4)^3 - 3$

Example 6: Let $f(x) = -x^3 + 3x^2 + 2$, Describe g as a transformation of f.

a. $g(x) = x^3 + 3x^2 + 2$

b. $g(x) = x^3 - 3x^2 - 2$

Example 7: For $f(x) = x^3 + 2$, describe the transformation.

a. $i\,(x) = \frac{1}{27}x^3 + 2$

b. $h(x) = 3(x + 1)^3 + 6$

Algebra 2, Polynomial Chapter

Section 11 Curve Fitting with Polynomial Models

Goal: Use technology to find polynomial models for a given set of data.

To create a mathematical model for data, you will to determine what type of function is most appropriate. Finite differences can be used to identify the degree of any polynomial data.

Finite Differences of Polynomials

Function Type	Degree	Constant Finite Differences
Linear	1	First
Quadratic	2	Second
Cubic	3	Third
Quartic	4	Fourth
Quintic	5	Fifth

Often, real-world data can be too irregular to use the finite differences method to find a polynomial function that fits perfectly. In these situations, you can use the **regression feature** of your graphing calculator. Remember that the closer the R^2-value is to 1, the better the function fits the data. You should attempt **linear** $(ax + b)$, **quadratic**, **cubic**, and **quartic** regressions.

Ex. 1: Use the regression feature of your graphing calculator to find a polynomial function that best describes the data.

1. Enter in the data in L_1 and L_2.
2. Perform regressions on the data (beginning with linear and progressing to higher degrees) to determine the R^2-value that is closest to 1. Be sure to store the result of each regression into y_1.
3. Find the coefficient values located in y_1 and write the polynomial function.

a.

x	12	15	18	21	24	27
y	3	23	29	29	31	43

b.

x	*y*	*x*	*y*
−6	−9	3	41
−3	16	6	78
0	26	9	151

c. The table below shows the gas consumption of a compact car driven a constant distance at various speeds.

Speed	25	30	35	40	45	50	55	60
Gas (gal)	23.8	25	25.2	25	25.4	27	30.6	37

Ex. 3. Use a polynomial model to estimate the value of the index in 1999.

Year	1994	1995	1996	2000	2003	2004
Price ($)	3754	3835	5117	11,497	8342	10,454

Step 1: Make a scatter plot with x representing the number of years since 1994. Use the regression feature to check R^2-values.
Step 2: Write a polynomial function.
Step 3: Find the value that corresponds to the x-value needed.